PUTNAM PRACTICE SET 9

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Problem 1. Find all real numbers a, b, c with the property that the equation

$$x^3 + ax^2 + bx + c = 0$$

has 3 real roots r_1, r_2, r_3 (not necessarily distinct) with the property that the equation

$$x^3 + a^3x^2 + b^3x + c^3 = 0$$

has the roots r_1^3, r_2^3, r_3^3.

Problem 2. Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-constant function with the property that for each $x, y > 0$ we have that $f(xy) = f(x)f(y)$. Find two functions $g, h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying the properties:

- $h\left(\frac{x}{y}\right) = h(x)h(y) - g(x)g(y)$ for each $x, y > 0$; and
- $h(x) + g(x) = f(x)$ for each $x > 0$.

Problem 3. Let $x, y, z \in \mathbb{N}$ such that $xy - z^2 = 1$. Prove that there exist nonnegative integers a, b, c, d such that $x = a^2 + b^2$, $y = c^2 + d^2$ and $z = ac + bd$.

Problem 4. Let $P, Q \in \mathbb{R}[x, y]$ be polynomials satisfying the following properties:

(A) for each $y_0 \in \mathbb{R}_{\geq 0}$, the functions $x \mapsto P(x, y_0)$ and $x \mapsto Q(x, y_0)$ are strictly increasing;
(B) for each $x_0 \in \mathbb{R}_{\geq 0}$, the function $y \mapsto P(x_0, y)$ is strictly increasing, while the function $y \mapsto Q(x_0, y)$ is strictly decreasing; and
(C) $P(x, 0) = Q(x, 0)$ for each $x \in \mathbb{R}_{\geq 0}$ and also, $P(0, 0) = 0$.

Prove the following:

(1) for each real numbers $0 \leq b \leq a$, there exists a unique pair (x_0, y_0) of nonnegative real numbers with the property that $P(x_0, y_0) = a$ and $Q(x_0, y_0) = b$.
(2) if $0 \leq a < b$, then there exist no nonnegative real numbers x_0 and y_0 such that $P(x_0, y_0) = a$ and $Q(x_0, y_0) = b$.